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Bond Index Group
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J.P. Morgan Yield Curve Model

1 Introduction

This publication introduces the J.P. Morgan Yield Curve Model for government bond markets. A yield curve approximates the tradeoff between risk and return for a market of similar securities. If the appropriate yield curve is specified and calculated, one may determine those bonds which are trading cheap or dear relative to the rest of the securities in the market. Our model has proved to be a valuable tool in selecting these opportunities over the past year and a half for the markets tested: West Germany, Canada, and France.

Since government bond markets usually form a very liquid, homogeneous credit market, their yield curves serve as benchmarks for most non-government bond markets.

The theories underlying our model were developed in the 1970s and early 1980s and have been used extensively since then, particularly in the U.S. markets. What is new, is the effort to apply a sound model uniformly across different government bond markets. This leads to a better understanding of the relationships between the markets and a more consistent evaluation of risks across markets.

The remainder of this publication:

- describes the model
- interprets a sample Cheap/Dear report
- summarizes the historical model performance, and
- provides a complete mathematical specification of the model.

2 Model description

Our goal is to best describe the risk/return characteristics of a market, where *market* is a homogeneous group of bonds (e.g., the U.S. Government Bond market). The risk of a single bond measures how its current price moves when market conditions change; its return measures the current earnings rate or internal rate of return. The most widely used concept to describe the risk/return characteristic of a market is the *yield to maturity* (YTM). The traditional yield curve shows the YTM for a group of comparable bonds against their respective terms to maturity. Unfortunately, this representation of the yield curve is flawed for coupon-bearing securities. A coupon bond is a series of future cash flows and the valuation of a coupon bond should take into consideration the timing of all the cash flows. The YTM approach wrongly discounts each cash flow by the same rate.

In the early 1970s an entirely different approach was developed: instead of looking at a bond as an inseparable set of cash flows and calculating its internal rate of return, the bond is decomposed and each cash flow is evaluated separately as a zero coupon bond. The relationship between the rates on the individual cash flows and the times to the cash flows is the *zero coupon curve*. The value of a bond equals the sum of the values of the *zero coupon bond* evaluated at the market's zero coupon curve. This curve is generally not observable and must be derived from the prices of the coupon securities. Thus the market's zero coupon curve became defined as: the yield curve which, when applied to the individual cash flows of each bond in the market and aggregated, best approximates the observed price of each bond in the market. The zero coupon curve is a cleaner and more useful yardstick for measuring the risk/return characteristics of a market .

The J.P. Morgan Yield Curve Model is based on this zero coupon concept. The remainder of this section describes how we calculate the zero coupon curve and how it is applied to estimate *fair* or *theoretical* prices for each bond in a homogeneous group of bonds. A detailed description of all the math is provided in the last section of this publication.

The present value or price of a zero coupon bond is :

$$(1) \qquad ZP_k = \frac{C_k}{(1 + Z_k)^{t_k}}$$

where:
- t_k = the time to the kth cash flow measured in coupon periods
- ZP_k = the price of a zero coupon bond paying principal in t_k
- C_k = the amount paid in t_k
- Z_k = the zero coupon rate for t_k

Z_k is the rate at which an investment grows from now to t_k priced at current market conditions. The relationship between Z_k and t_k is the zero coupon curve, which shows how borrowers are currently compensating lenders for discount loans over different time horizons. This relationship is referred to as the *term structure of interest rates.*

A coupon bond is a simple bundle of zero coupon bonds and therefore the current price equals the sum of the present values of the future cash flows C_k:

$$(2) \qquad P = \sum_{k=1}^{K} \frac{C_k}{(1+Z_k)^{t_k}}$$

where:
- P = Gross Price; the clean price plus accrued interest
- k = 1, 2, ..., K; the counter for the separate future cash flows in bond j.
- K = the number of cash flows for the coupon bond including redemption

$1/(1+Z_k)^{t_k}$ is the number which equates the kth cash flow to its present value and is known as the "discount factor" for C_k. The discount factor represents how much an investor will pay now for $1 received at t_k. The relationship between the discount factors and the time to the cash flow is known as the discount function and is related to the term structure of interest rates by $1/(1+Z_k)^{t_k}$.

When investors purchase securities, they implicitly calculate the discount factors. For a zero coupon bond, the discount factor is simply P/100 where P is quoted per 100 nominal amount. For coupon bonds there are discount factors for each cash flow. The price determines the sum of the discount factors; however, the price does not determine the individual discount factors for each t_k. By calculating a linear regression on a market cross-section of known cash flows and observable prices, we may estimate the individual discount factors.

$$(3) \qquad MP_j = \sum_{k=1}^{K} C_{jk} D_k + \Delta_j$$

where:
- MP_j = gross market price
- C_{jk} = kth cash flow for bond j
- D_k = average discount factor for all j
- Δ_j = price difference or error term
- j = 1, 2, ..., J; J = the number of securities in the model

This technique introduces price differences Δ_j into the pricing equation. Such differences, or error terms, occur because this model cannot explain all the factors which influence the market price of an individual bond (e.g., liquidity, tax effects etc.). The analysis of these differences over time will provide valuable indications on whether a particular bond is relatively cheap, well-priced or relatively expensive at any point in time.

In order to estimate equation (3), we must model the individual discount factors. We know that the main determinant of the discount factors are the times to the cash flows. Because we do not know the precise form of this relationship, we express the discount function as an Ith degree polynomial in time. A polynomial approximates any regular function with small error.

(4) $$D_k = \alpha_0 + \alpha_1 t_k^1 + \alpha_2 t_k^2 + ... + \alpha_I t_k^I$$

where: I = the degree of the polynomial. The optimal I is statistically determined.

α_i = the coefficients of the model which describe how the time to the cash flow determines the discount factors.

By substituting D_k in equation (4) into equation (3), we may estimate the coefficients using linear regression.

(5) $$MP_j = \sum_{k=1}^{K} C_{jk} \sum_{i=0}^{I} \widehat{\alpha}_i t_k^i + \widehat{\Delta}_j$$

where: $\widehat{\alpha}_i$ = the estimated coefficients of the model

$\widehat{\Delta}_j$ = the estimated error term

This technique finds the alphas which minimize the sum of the squared differences (see the mathematical appendix for this solution). We then use the alphas to calculate the *theoretical*, *model*, or *synthetic* price with the following equation:

(6) $$TP_j = \sum_{k=1}^{K} C_{jk} \sum_{i=0}^{I} \widehat{\alpha}_i t_k^i$$

We see from equations (5) and (6) that the theoretical price of the bond is related to the market price by the error term in the regression: $MP = TP + \Delta$. The difference between the theoretical price and the market price (Δ) contains the components of the market price which are not explained by or correlated with the term structure. For example, in a government bond market where securities are relatively homogeneous and default-free, these differences usually arise due to coupon and liquidity effects, although certain coupon effects may also be correlated with the term structure (Japanese market). An investor may prefer high or low coupons for tax reasons. Also, investors will pay a premium for highly liquid bonds and require additional yield for illiquid bonds.

These factors change slowly over time and may be captured by calculating a historical average of the difference between the actual price and the model price, Mean (Δ_j), to measure the more permanent premium or discount the market assesses for each bond. The temporary component of Δ_j arises due to mispricing. The market is not always completely efficient in pricing bonds and this is reflected in the temporary component. By effectively measuring this latter component, we may identify buying or selling opportunities. This temporary component is measured by subtracting Δ_j from the its more permanent component Mean (Δ_j). We refer to this statistic, Mean (Δ_j)-Δ_j, as the J.P. Morgan Cheap/Dear statistic.

To estimate equation (6), we use data from the J.P. Morgan Government Bond Market Database. This source provides for each reasonably liquid bond in the major government bond markets:

- exact and verified cash flow information
- precise accrued interest calculations
- timely new issue information
- reliable price data from local offices
- monthly liquidity assignments by Morgan traders

The reliability of data is a critical factor for the successful development, testing and application of pricing models.

3 The Cheap/Dear Report

The page on the right shows an example of one of J.P. Morgan's daily Cheap/Dear Reports; in this instance, a report on the prices of the principal West German Government Bonds on September 15, 1989. The bonds contained in this report are those considered *regularly traded* by Morgan market makers in Frankfurt. This report illustrates a typical use of the yield curve model. The goal is to provide the user with an easily readable table highlighting which specific securities are cheap or dear relative to the curve.

The following is a description of the individual columns in the table which provides information on the 63 liquid German Government bonds in order of increasing remaining maturity.

Issues: The coupon rate, security type and maturity date are displayed to identify each security.

Duration: The duration of the security measures the bond's price sensitivity to small changes in its yield. The longer the duration, which is measured in years, the larger the price change for a given change in yield.

Yield

- **Actual:** This yield is the internal rate of return (IRR) on the bond. The IRR is the rate which equates the net present value of all the future cash flows to the actual price of the bond. This rate is often different from the yield quoted in the market.

- **Expected:** This is the yield that would prevail if the bond was trading at its model price plus the average price deviation over the last 30 business days. It assumes the bond is correctly priced according to our term structure model. This yield corresponds to the synthetic price of the bond adjusted for the 30-day mean of the actual-less-synthetic price difference (see statistics below).

Price

- **Actual:** This is the clean price of the security at the time given at the bottom of the report.

- **Synthetic:** The synthetic price is the theoretical price of the bond that we estimate with our model. It represents the price at which the bond would be trading if there were no premiums or discounts associated with the bond relative to all other bonds in the sample and if there were no mispricing.

- **Difference (Δ):** This is the difference between the actual price and the model price (actual price - model price). If positive (negative), this statistic measures the net premium (discount) of the bond including any mispricing.

Statistics

- **30 days mean:** This is the average deviation between the actual price and the model price over the last 30 business days. This statistic measures the more permanent component of the price difference. A positive (negative) number indicates a premium (discount) associated with the bond in question.

- **Cheap/Dear:** This statistic is the current deviation between the actual and model prices minus the historical deviation (Δ - 30 days mean). This number measures the temporary mispricing of the security. A positive(negative) Cheap/Dear statistic indicates that the bond is trading expensively (inexpensively).

- **Z-score:** This value is the Cheap/Dear statistic divided by the 30 day standard deviation of Δ. Dividing by the standard deviation adjusts the Cheap/Dear statistic such that they are all on the same scale in terms of volatility. As a general rule, a z-score with an absolute value greater than 1 is deemed significant and indicates that the price abberation has a high probability of reversing.

The back page of this publication shows the yield curves which are associated with these statistics.

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September 15, 1989

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Government Bond Markets
Daily Cheap Dear Analysis

JPMorgan

Issues	Dur	Yield Actual	Yield Exp.	Price Actual	Price Synth	Price Diff:Δ	30days Mean(Δ)	Statistics on Δ: (Actual - Model) Price — Cheap Dear: Δ - Mean(Δ)	Z-score: (Δ-Mean(Δ))/Sigma(Δ)
1 4.875% Kassen Jun 91	1.70	7.30	7.27	96.10	96.20	-0.10	-0.06	-0.04	-1.24
2 5.250% Kassen Dec 91	2.65	7.32	7.30	95.85	96.07	-0.22	-0.18	-0.04	-0.93
3 5.000% Kassen Jan 92	2.19	7.27	7.25	95.20	95.35	-0.15	-0.10	-0.04	-1.14
4 5.250% Kassen Mar 92	2.35	7.23	7.25	95.55	95.63	-0.08	-0.14	0.05	1.40
5 6.000% Kassen Jun 92	2.59	7.23	7.24	96.95	97.07	-0.12	-0.14	0.02	0.48
6 5.375% Kassen Feb 93	3.11	7.21	7.21	94.50	94.67	-0.17	-0.18	0.02	0.28
7 5.000% Kassen Apr 93	3.30	7.23	7.21	93.10	93.34	-0.24	-0.18	-0.06	-0.42
8 5.375% Kassen May 93	3.36	7.23	7.23	94.15	94.39	-0.24	-0.23	-0.01	-0.17
9 6.250% Kassen Sep 93	3.44	7.21	7.20	96.70	96.96	-0.26	-0.23	-0.03	-0.58
10 5.500% Bobl May 91 #63	1.62	7.18	7.22	97.40	97.31	0.09	0.03	0.06	1.14
11 5.500% Bobl Dec 91 #67	2.10	7.16	7.22	96.60	96.51	0.09	-0.04	0.13	1.76
12 5.250% Bobl Feb 92 #68	2.27	7.09	7.11	95.95	95.73	0.22	0.19	0.03	0.53
13 5.500% Bobl Aug 92 #71	2.76	7.10	7.12	95.90	95.68	0.22	0.18	0.03	0.66
14 6.000% Bobl Oct 92 #73	2.75	7.09	7.10	97.00	96.82	0.18	0.16	0.03	0.42
15 5.500% Bobl Nov 92 #74	2.86	7.08	7.09	95.55	95.36	0.19	0.17	0.03	0.42
16 5.250% Bobl Feb 93 #75	3.13	7.09	7.00	94.45	94.29	0.16	0.41	-0.25	-0.34
17 5.500% Bobl May 93 #78	3.36	7.09	7.09	94.95	94.78	0.17	0.17	-0.01	-0.07
18 6.000% Bobl Aug 93 #79	3.58	7.09	7.08	96.35	96.18	0.17	0.21	-0.04	-0.34
19 5.750% Bobl Oct 93 #80	3.56	7.09	7.06	95.30	95.19	0.11	0.23	-0.12	-1.22
20 6.500% Bobl Jan 94 #83	3.76	7.07	7.06	97.80	97.62	0.18	0.20	-0.02	-0.18
21 6.750% Bobl Apr 94 #84	3.99	7.08	7.12	98.65	98.50	0.15			
22 8.250% Bund Jan 94	3.59	7.17	7.17	103.80	103.85	-0.05	-0.06	0.01	0.19
23 8.250% Bund Feb 94	3.68	7.17	7.15	103.85	103.94	-0.09	-0.05	-0.05	-0.53
8.000% Bund Mar 94	3.82	7.15	7.15	103.05	103.18	-0.13	-0.13		0.03
8.250% Bund Jun 94	4.06	7.15	7.14	104.20	104.34	-0.14	-0.10	-0.04	-0.79
26 8.250% Bund Jul 94	4.15	7.14	7.15	104.35	104.43	-0.08	-0.11	0.03	0.64
27 8.250% Bund Aug 94	4.24	7.13	7.13	104.45	104.52	-0.07	-0.08	0.01	0.13
28 7.500% Bund Oct 94	4.14	7.14	7.14	101.45	101.55	-0.10	-0.09	-0.01	-0.43
29 7.000% Bund Dec 94	4.35	7.05	7.09	99.70	99.45	0.25	0.08	0.16	1.85
30 7.000% Bund Jan 95	4.44	7.07	7.10	99.60	99.43	0.17	0.06	0.11	1.34
31 7.250% Bund Feb 95	4.50	7.11	7.11	100.50	100.51	-0.01	0.01	-0.02	-0.52
32 7.625% Bund Mar 95	4.55	7.09	7.09	102.20	102.22	-0.02	-0.02	-0.00	-0.02
33 7.500% Bund Apr 95	4.64	7.12	7.11	101.60	101.69	-0.09	-0.04	-0.05	-1.14
34 7.250% Bund May 95	4.75	7.11	7.11	100.55	100.61	-0.06	-0.04	-0.02	-0.49
35 7.000% Bund Jun 95	4.85	7.05	7.08	99.65	99.48	0.17	0.04	0.14	2.03
36 6.750% Bund Jul 95	4.96	7.10	7.09	98.30	98.34	-0.04	0.02	-0.06	-1.75
37 6.500% Bund Oct 95	4.91	7.10	7.09	97.05	97.11	-0.06	0.01	-0.07	-1.53
38 6.375% Bund Jan 96	5.18	7.08	7.06	96.40	96.38	0.02	0.07	-0.05	-0.81
39 6.375% Bund Feb 96	5.26	7.09	7.09	96.30	96.34	-0.04	0.04	-0.08	-1.29
40 5.750% Bund Jun 96	5.67	7.06	7.06	93.00	93.01	-0.01	-0.04	0.03	0.43
41 6.500% Bund Dec 96	5.71	7.04	7.04	96.85	96.81	0.04	0.05	-0.02	-0.30
42 6.125% Bund Jan 97	5.85	7.03	7.04	94.80	94.69	0.11	0.05	0.06	1.22
43 5.750% Bund Feb 97	5.99	7.03	7.05	92.65	92.55	0.10	-0.02	0.12	2.02
44 6.000% Bund Mar 97	6.03	7.05	7.05	93.95	93.96	-0.01	-0.02	0.01	0.20
45 5.500% Bund May 97	6.28	7.06	7.06	90.90	91.01	-0.11	-0.11		0.01
46 6.125% Bund Jul 97	6.35	7.06	7.03	94.45	94.57	-0.12		-0.12	-2.87
47 6.375% Bund Aug 97	6.40	7.05	7.03	95.95	96.03	-0.08	0.01	-0.09	-2.03
48 6.750% Bund Sep 97	6.02	7.01	7.00	98.35	98.26	0.09	0.15	-0.06	-0.81
49 6.375% Bund Oct 97	6.16	7.01	7.02	96.10	96.01	0.09	0.01	0.08	1.49
50 6.375% Bund Jan 98	6.42	7.01	7.01	95.90	95.94	-0.04	-0.02	-0.02	-0.40
51 6.250% Bund Feb 98	6.52	7.02	7.03	95.10	95.16	-0.06	-0.08	0.02	0.29
52 6.000% Bund Apr 98	6.73	6.99	6.99	93.60	93.61	-0.01	-0.03	0.01	0.22
53 6.500% Bund May 98	6.73	6.99	6.99	96.75	96.76	-0.01	-0.04	0.02	0.56
54 6.750% Bund Jul 98	6.85	6.99	6.99	98.40	98.41	-0.01	0.02	-0.02	-0.47
55 6.750% Bund Aug 98	6.93	6.99	6.98	98.40	98.44	-0.04	-0.01	-0.03	-0.71
56 6.000% Bund Oct 98	6.80	6.97	6.97	93.60	93.59	0.01	-0.01	0.02	0.39
57 6.375% Bund Nov 98	6.81	6.97	6.97	96.00	96.04	-0.04	-0.04		0.15
58 6.375% Bund Dec 98	6.89	6.95	6.95	96.10	96.05	0.05	0.01	0.04	1.25
59 6.500% Bund Jan 99	6.91	6.94	6.94	96.85	96.86	-0.01	0.02	-0.02	-0.64
60 6.750% Bund Jan 99	6.91	6.94	6.94	98.55	98.56	-0.01	0.01	-0.01	-0.25
61 7.000% Bund Feb 99	6.96	6.91	6.92	100.45	100.26	0.19	0.08	0.10	2.05
62 7.000% Bund Apr 99	7.11	6.91	6.92	100.45	100.40	0.05	0.02	0.03	0.51
63 6.750% Bund Jun 99	7.32	6.93	6.91	98.70	98.83	-0.13			

|z| > 1 is significant

Price Source Frankfurt Office/Rate Fixings, 11 am Friday September 15, 1989

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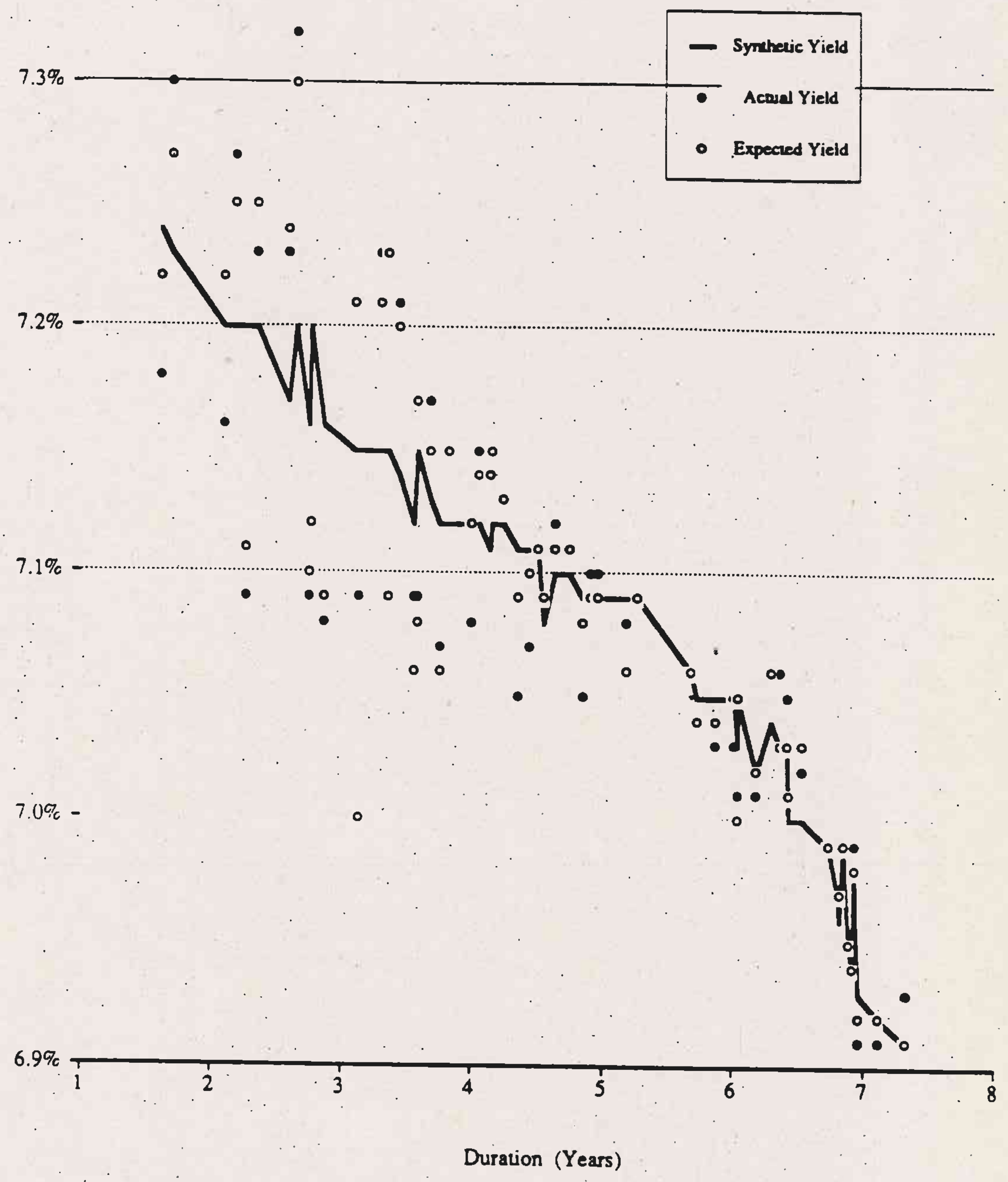
German Government Bonds

Cheap/Dear Analysis

September 15, 1989



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4 Historical model performance

A model that attempts to recommend cheap and dear securities is worthless unless tested historically. Our historical analyses show that on average the discount function model adds value for investors planning to transact in the market. This section describes the profitability results for West Germany, Canada, and France.

We calculated the discount function on every trade day from January 1, 1988 to September 15, 1989. We stored the coefficients, the price differences, the cheap/dear statistics and the z-scores. On average, the model explained over 99% of the differences between security prices within each market. We used the statistics to recommend trades and then tracked the performance of the trades to see if they were profitable. In order to make consistent decisions over time, we decided on a set of "rules" to determine the trades:

a) We buy or sell outright 1,000,000 (nominal amount in local currency) of a bond based on its z-score and cheap/dear statistic.

   1 Buy a bond if the z-score is less than -2 and the cheap/dear statistic is less than -0.1

   2 Sell a bond if the z-score is more than 2 and the cheap/dear statistic is more than 0.1

b) We then take an offsetting position (duration weighted trade) with another bond whose:

   1 Maturity is within six months

   2 Coupon is within 200 basis points

   3 Z-score is furthest in the opposite direction and at least 2 standard deviations away from the initial bond

c) Finally, we reverse the trade if the difference in the z-score between the two bonds reduces by 50%.

For each trade we calculated the profit/loss from changes in price and accrued interest. The following table summarizes these results.

	Germany	Canada	France
Total number of trades	1049	369	126
Average profit per switch	DM828	C$863	FF1169
Percent profit per switch	5.8%	10.8%	22.0%
Percent of trades that were profitable	75%	70%	74%
Average length of the trades (days)	5	2.5	2

Almost three-quarters of the trades were profitable using the Yield Curve Model for security selection. These profits do not account for transactions costs but represent the implied profit of selecting the cheapest security over the most expensive maturity. The results indicate that on average the prices for those securities deemed cheap rise over the next few days and fall for those deemed dear. Due to the transactions costs, we do not recommend this model for high frequency trading; however, we do feel it offers additional yield pick-up for investors already interested in investing in the market or divesting out of the market.

For a more detailed look at the profitability, the table on the adjacent page shows the number of trades broken down by level of profitability, maturity, and by the cheap/dear statistic of the initial bond.

Profitability of the Yield Curve Model

(Number of trades in each category, from Jan 1, 1987 to Sep 15, 1989)

West Germany

Profit (DM)	(3000)-(2000)	(2000)-(1000)	(1000)-0	0-1000	1000-2000	2000-3000	3000-4000	>4000	Total
by Maturity									
< 5 yrs	0	13	70	156	57	4	3	14	317
5-7 yrs	0	10	55	146	82	26	3	7	329
7+ yrs	1	11	102	156	87	28	7	11	403
Total	1	34	227	458	226	58	13	32	1049
by C/D Stat									
2.0-2.2	1	22	143	300	147	35	9	19	676
2.2-2.4	0	4	30	78	33	8	1	1	155
2.4-2.6	0	2	23	33	14	6	1	1	80
2.6-2.8	0	1	14	26	13	3	1	0	58
2.8-3.0	0	1	13	11	8	3	0	2	38
3.0 +	0	4	4	10	11	3	1	9	42
Total	1	34	227	458	226	58	13	32	1049

Canada

Profit (C$)	(3000)-(2000)	(2000)-(1000)	(1000)-0	0-1000	1000-2000	2000-3000	3000-4000	>4000	Total
by Maturity									
< 2	0	0	8	10	4	3	1	3	29
2-5 yrs	6	4	56	56	34	17	10	7	190
5-7 yrs	0	0	13	10	10	6	3	4	46
7-10 yrs	0	1	14	23	16	7	2	11	74
10+ yrs	0	2	6	14	4	1	0	3	30
Total	6	7	97	113	68	34	16	28	369
by C/D Stat									
2.0-2.2	3	5	65	81	44	23	10	21	252
2.2-2.4	0	1	13	15	6	2	1	1	39
2.4-2.6	1	1	7	7	7	3	2	2	30
2.6-2.8	0	0	7	2	2	2	1	1	15
2.8-3.0	1	0	1	2	3	2	0	0	9
3.0 +	1	0	4	6	6	2	2	3	24
Total	6	7	97	113	68	34	16	28	369

France

Profit (Ffr)	(3000)-(2000)	(2000)-(1000)	(1000)-0	0-1000	1000-2000	2000-3000	3000-4000	>4000	Total
by Maturity									
< 2	0	0	0	1	3	0	0	0	4
2-5 yrs	0	0	3	8	3	3	0	0	16
5-7 yrs	1	3	9	16	14	3	3	5	54
7-10 yrs	0	2	17	16	8	6	1	2	52
10+ yrs	0	0	0	0	0	0	0	0	0
Total	1	5	29	41	28	11	4	7	126
by C/D Stat									
2.0-2.2	1	2	20	28	19	9	3	3	85
2.2-2.4	0	2	3	5	2	0	1	0	13
2.4-2.6	0	1	2	5	1	2	0	2	13
2.6-2.8	0	0	3	3	3	0	0	0	9
2.8-3.0	0	0	0	0	1	0	0	0	1
3.0 +	0	0	1	0	2	0	0	2	5
Total	1	5	29	41	28	11	4	7	126

5 Detail specification

This section provides in detail the mathematics underlying the JP Morgan Yield Curve Model . The first part describes how we estimate the theoretical price of a bond. The second part shows how implied yield curves are calculated from an estimated discount function.

5.1 Theoretical Prices

The theoretical price of a bond, TP_j, is the sum of the present values of its future cash flows:

(1) $$TP_j = \sum_{k=1}^{K} C_{jk} D_k$$

The market price MP_j, which is known, equals the theoretical price plus an error term Δ_j:

(2) $$MP_j = TP_j + \Delta_j = \sum_{k=1}^{K} C_{jk} D_k + \Delta_j$$

• **Discount function:** the J.P. Morgan Yield Curve Model assumes the discount function D_k is an Ith degree polynomial in time:

(3) $$D_k = \alpha_0 + \alpha_1 t_k + \alpha_2 t_k^2 + \alpha_3 t_k^3 + \ldots + \alpha_I t_k^I = \sum_{i=0}^{I} \alpha_i t_k^i$$

Equation (3) shows the discount function for a cash flow received in t_k. This specification assumes that the market discounts any future cash flow with a discount function which is the same for all bonds; i.e., $D_{jk} = D_k$ for all j. The discount function D_k should obey the following three conditions:

When $t_k = 0$, the discount function equals 1; therefore, we set $\alpha_0 = 1$.

As t_k decreases and approaches 1 day, the discount function should approach the discount factor on the overnight rate ON; therefore, we set $\alpha_1 = -\ln[1+ON]$. This result is derived in the section 5.2 on the zero coupon yield curve.

For any t the discount function must be monotonically decreasing; $D_t > D_s$ where $t > s$, and it must be positive; $D_t > 0$

By substituting D_k from equation (3) into equation (2) we obtain:

(4) $$MP_j = \sum_{k=1}^{K} C_{jk} \sum_{i=0}^{I} \alpha_i t_k^i + \Delta_j$$

which can be rearranged to:

(5) $$MP_j = \sum_{i=0}^{I} \alpha_i \sum_{k=1}^{K} C_{jk} t_k^i + \Delta_j$$

The discount function which best explains the current market prices MP_j as a function of future cash flows C_{jk} is the one which minimizes the sum of squared residuals Δ_j. That is solve for the α_i's (for $i>1$) in order to:

(6) $$\text{minimize} \sum_{j=1}^{J} (\Delta_j)^2$$

Since we constrain α_0 and α_1, we may rewrite equation (5):

(7) $$MP_j - \alpha_0 \sum_{k=1}^{K} C_{jk} - \alpha_1 \sum_{k=1}^{K} C_{jk} t_k = \sum_{i=2}^{I} \alpha_i \sum_{k=1}^{K} C_{jk} t_k^i + \Delta_j$$

• **Estimation:** Equation (7) describes a standard linear regression problem. The residuals Δ_j are:

(8) $$\Delta_j = MP_j - \alpha_0 \sum_{k=1}^{K} C_{jk} - \alpha_1 \sum_{k=1}^{K} C_{jk} t_k - \sum_{i=2}^{I} \alpha_i \sum_{k=1}^{K} C_{jk} t_k^i$$

or in vector notation: $\Delta = \mathbf{MP} - \mathbf{G} - \mathbf{Z}\beta$

where: Δ = column vector of length J with the elements Δ_j

MP = column vector of length J with the elements MP_j

G = column vector of length J with the elements:

$$\alpha_o \sum_{k=1}^{K} C_{jk} + \alpha_1 \sum_{k=1}^{K} C_{jk} t_k$$

Z = matrix with J rows and I-1 columns containing the elements:

$$Z_{ji} = \sum_{k=1}^{K} C_{jk} t_k^{i+1}$$

β = row vector of length I-1 with the elements $[\alpha_2, \alpha_3, ..., \alpha_I]$, the unknowns.

Note that vectors and matrices are in bold. Equation (6) may be rewritten as:

$$\text{(9)} \quad \text{minimize} \sum_{j=1}^{J} (\mathbf{F} - \mathbf{Z}\boldsymbol{\beta})^2$$

where: **F** = **MP** - **G**

The estimate for β is found by using ordinary least squares (OLS). The OLS estimator is defined as:

$$\text{(10)} \quad \hat{\boldsymbol{\beta}} = (\mathbf{Z}^T\mathbf{Z})^{-1}\mathbf{Z}^T\mathbf{F}$$

where: $\hat{\boldsymbol{\beta}}$ = row vector of length I-1 with the elements $[\hat{\alpha}_2, \hat{\alpha}_3, ..., \hat{\alpha}_I]$

$\mathbf{Z}^T$ = the transpose of **Z**

• **Estimated prices:** Given estimates for the parameters of the model, we may calculate the estimated theoretical prices of the bonds.

$$\text{(11)} \quad \widehat{\mathbf{TP}} = \mathbf{G} + \mathbf{Z}\hat{\boldsymbol{\beta}}$$

where: $\widehat{\mathbf{TP}}$ = column vector of length J containing the estimated theoretical prices

and each element $\widehat{TP}_J$ in $\widehat{\mathbf{TP}}$ is:

$$\text{(12)} \quad \widehat{TP}_j = \alpha_o \sum_{k=1}^{K} C_{jk} + \alpha_1 \sum_{k=1}^{K} C_{jk} t_k + \sum_{i=2}^{I} \hat{\alpha}_i \cdot \sum_{k=1}^{K} C_{jk} t_k^i$$

Since accrued interest A_j is the same for the theoretical and market prices, we may subtract it from the theoretical gross price $\widehat{TP}_j$ to obtain the theoretical clean price:

$$\text{(13)} \quad \widehat{TCP}_j = \widehat{TP}_j - A_j$$

$\widehat{TCP}_J$ is referred to as the "synthetic price" in the Cheap/Dear report.

5.2 Yield curves

This section derives two standard yield curves from the discount function defined in the previous section: the synthetic zero coupon yield curve and the synthetic par yield curve. The term synthetic is used to indicate that the yield curves are based on a theoretical model (which was used to calculate the discount function).

• **Zero Coupon Yield Curve:** The discount function equates the future value FV of a cash flow C_k to its present value PV; whereas the zero coupon yield allows the present value PV to be compounded into a future value FV:

$$\text{(14)} \quad PV(C_k) = FV(C_k)\, D_k$$

$$FV(C_k) = PV(C_k)\,(1 + Z_k)^{t_k}$$

Z_k is the rate at which the present value of C_k grows from now to t_k. Thus, we can solve for D_k

$$\text{(15)} \quad D_k = (1 + Z_k)^{-t_k}$$

This equation may be expressed such that D is a function of time. This allows U.S. to express the relationship between D and Z in continuous time.

(16) $$D(t) = (1 + Z(t))^{-t}$$

This equation rearranged leads to the synthetic zero coupon yield curve Z(t) in terms of the discount function D(t) defined in equation (3)

(17) $$Z(t) = D(t)^{-1/t} - 1$$

Since we estimate the discount function we may then calculate the synthetic or estimated zero coupon curve:

(18) $$\widehat{Z}(t) = \widehat{D}(t)^{-1/t} - 1$$

Next, we need to prove the constraint that $\alpha_1 = \ln(1+ON)$ postulated in the conditions of equation (3). As t approaches zero, the zero coupon yield approaches the instantaneous rate of interest. We use the overnight rate, ON, as a proxy for the instantaneous rate. The contraint can be derived by observing how the discount function behaves as t approaches zero. The first step is to linearize equation (18) by taking logarithms:

(19) $$\ln(1 + Z(t)) = \frac{-\ln D(t)}{t}$$

Using L'Hopital's rule we can rewrite equation (19) as:

(20) $$\ln(1 + Z(t)) = \frac{-D'(t)}{D(t)}$$

where D'(t) is the time derivative of D(t). Substituting in for D'(t) and D(t) we obtain:

(21) $$\ln(1 + Z(t)) = \frac{-\alpha_1 - 2\alpha_2 t - ... - I\alpha_I t^{I-1}}{\alpha_0 + \alpha_1 t + ... + \alpha_I t^I}$$

Taking the limit of both sides of equation (21) as t approaches zero gives:

(22) $$\lim_{t \to 0} \ln(1 + Z(t)) = \frac{-\alpha_1}{\alpha_0}$$

As stated before, as t approaches zero, Z(t) approaches the overnight rate, ON. Also, we know from the first condition of equation (3) that $\alpha_0 = 1$.

Thus:

(23) $$\ln(1 + ON) = -\alpha_1$$

• **Par Yield Curve**: The synthetic par yield curve is a theoretical construct; it represents the yields of fictitious bullet bonds with fixed coupons which trade at par. These fictitious bonds are referred to as *Par Bonds* or *Current Coupon Bonds*. For simplicity, we only calculate yields for par bonds on coupon dates. In this case, the sum of the discounted future cash flows is 100:

(24) $$\sum_{k=1}^{K} C_k D_k = 100$$

Also by definition, the future cash flows consist only of a constant coupon CP ($CP = C_k$ for all k) and a final payment of 100 at maturity. Therefore equation (24) can be rewritten as:

(25) $$CP\left(\sum_{k=1}^{K} D_k\right) + 100D_K = 100$$

By definition the synthetic par yield with K coupons is equivalent to the coupon rate, PY = CP. From equation (27) we can calculate CP and therefore the par yield PY:

(26) $$PY = \frac{100 - (100D_K)}{\sum_{k=1}^{K} D_k}$$

Finally, since we estimate the discount function we may solve for the estimated par yield curve:

(27) $$\widehat{PY} = \frac{100\left(1 - \widehat{D}_K\right)}{\sum_{k=1}^{K} \widehat{D}_k}$$

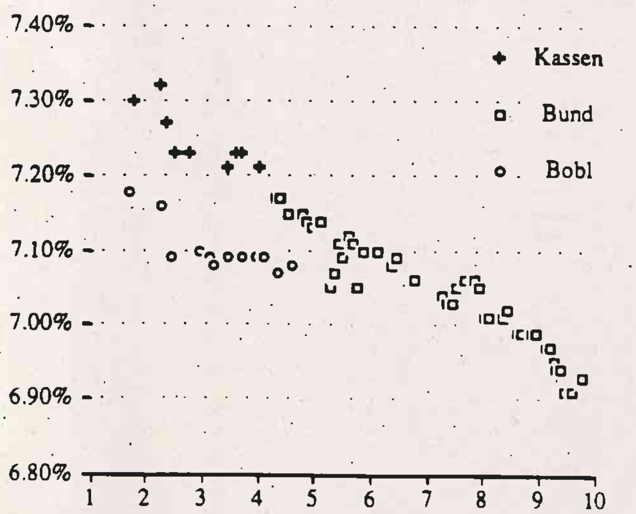
• **Example:** On the back page of this publication we illustrate the yield curves presented in this section. The graphs were derived from the prices of the 63 liquid government bonds in West Germany on August 1, 1989 listed in the report on page 5.

This model was developed jointly by Eric Boulot in Paris, Mark Alexandridis and John Lobley in New York, Mike Wilson in London and David Tse in Tokyo.

Yield Curves for the German Government Bond Market

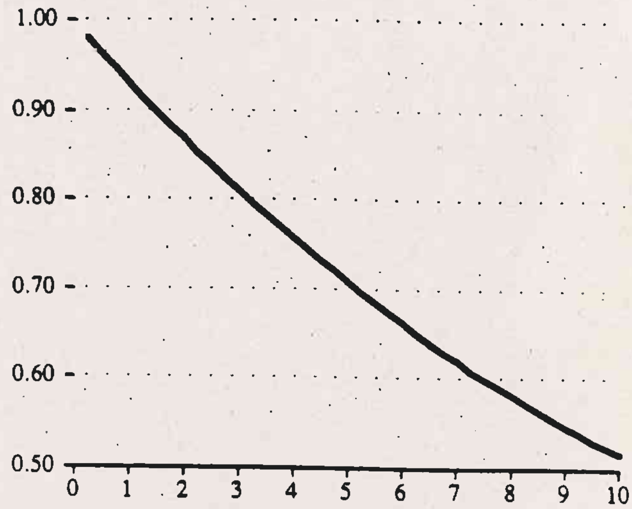
Fixing September 15, 1989

(1) Yield to maturity



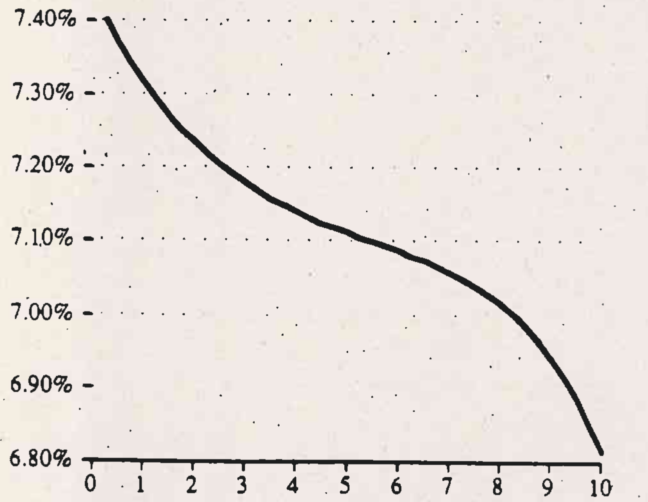
The yield to maturity for each of the 63 liquid bonds in the West German government bond market graphed against time to maturity.

(2) Discount function



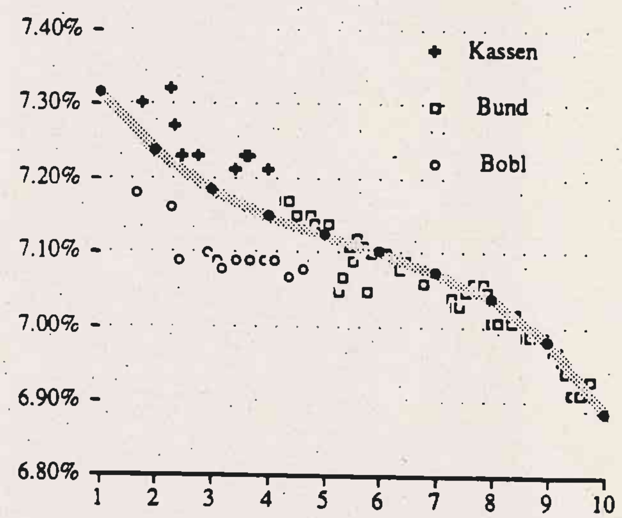
The discount function, derived from the prices of the 63 liquid coupon bonds, shows how the discount factors decrease as the time to cash flow increases. Each discount factor represents the value of 1DM received in the number of years shown on the horizontal axis.

(3) Synthetic zero coupon curve



The zero copon curve shows the relationship between the yield on a zero coupon bond and its term to maturity. This curve is not observable in the West German government bond market; it is derived from the discount function.

(4) Synthetic par curve



The synthetic par yield curve is overlaid on the yields to maturity graph shown above. The synthetic par yield curve shows the coupon level of bonds that would, under current market conditions be expected to trade at par, or 100. It is derived from the synthetic zero coupon curve.

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